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$\ell \times \ell'$-RAUZY GRAPHS FOR INFINITE ARRAYS

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ABSTRACT. Word representable graphs is a branch of study in combinatorics of words. One such word representable graph is $\ell \times \ell'$-Rauzy graph. Some structural properties of $\ell \times \ell'$-Rauzy graphs of order $k \times k'$ for infinite periodic arrays are studied such as the number of components in it and structure of each component in it. Given some larger values of k, k', the structure of the $\ell \times \ell'$-Rauzy graphs of order $k \times k'$ for infinite periodic arrays can be studied from a lower order $m \times n$, which is the size of the primitive root of the infinite periodic array. It is proved that the $\ell \times \ell'$-Rauzy graph for the infinite Fibonacci array is strongly connected.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 05C20, 05C40.

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1. INTRODUCTION

Graphs representing the words/languages are one of the important studies in combinatorics of words as well as theoretical computer science. Some of the word representable graphs are de Bruijn graphs, Rauzy graphs, half range Rauzy graphs and ℓ-Rauzy graphs. In this article, we define and study the $\ell \times \ell'$-Rauzy graphs for arrays.

Let $\Sigma = \{a_1, \cdots, a_n\}$ be an alphabet, and Σ^k be the set of all one-dimensional words of length k that are formed by the symbols from Σ. A word u of length k formed from Σ can be written as $u = u_1 \cdots u_k$ or $u = u[1, k]$.

A de Bruijn graph G of order k for an alphabet Σ is a directed graph with the vertex set $V(G) = \Sigma^k$ and for $u, v \in V(G)$, $e = uv$ forms an arc iff $u_2u_3 \cdots u_k = v_1v_2 \cdots v_{k-1}$. In [7], de Bruijn proves that for any de Bruijn graph of order k with 2^k vertices, there are $2^{2^{k-1}-k}$ different Hamiltonian cycles.

A set $F \subseteq \Sigma^*$, is said to be a factorial language if it contains all the factors/subwords of its words. Let $F(k) = F \cap \Sigma^k$. F_w is the set of all factors of w, and $F_w(m)$ is the set of all factors of w of length m.

A Rauzy graph of order k for a factorial language F, is a directed graph with the vertex set $V = F(k)$ and for any two vertices u and v, $e = uv$ forms an arc iff $u_2u_3 \cdots u_k = v_1v_2 \cdots v_{k-1}$ and $uv_k \in F(k+1)$. Rauzy graphs are widely used in finding the complexity of words of finite lengths. In 1991, Arnoux and Rauzy investigated the sequences with complexity $2n + 1$. G. Rote in [12] went one step further to Arnoux and Rauzy by constructing the sequences with complexity $2n$ using Rauzy graphs.

In [10], the authors introduced a special variant of Rauzy graphs by altering the sharing length $k - 1$ to $\frac{k}{2}$ (or $\frac{k-1}{2}, \frac{k+1}{2}$ if k is odd) and called it as half range

Rauzy graphs. The idea of altering the sharing length to half the length of the word represented by vertex was inspired by Adleman in [1].

A half range Rauzy graph of order k for a factorial language F, is a directed graph with the vertex set $V = F(k)$ and for any two vertices u and v, $e = uv$ forms an arc iff $u[\frac{|u|}{2}+1, |u|] = v[1, \frac{|v|}{2}]$ and the concatenated word uv is again a factor of the factorial language F, denoted by $\mathbb{HR}_F(k)$.

In [11], the authors developed another directed graph from the half range Rauzy graphs and named it as ℓ-Rauzy graphs. In an ℓ-Rauzy graph of order k, the authors generalize the sharing length to $1 \leq \ell < k$ among the words represented by the vertices to form an arc.

An ℓ-Rauzy graph of order k for a factorial language F, is a directed graph with the vertex set $V = F(k)$ and for any two vertices u and v, $e = uv$ forms an arc iff $u[|u| - \ell + 1, |u|] = v[1, \ell]$ and $u[1, |u|]v[|v| - \ell + 1, |v|] \in F(2k - \ell)$, denoted by ℓ-$\mathbb{R}_F(k)$. The ℓ-Rauzy graph is the generalisation of Rauzy graphs and half range Rauzy graphs.

Periodicity in words play a vital role in solving string matching problems. The periodicity in two-dimensional words is not as simple as in one-dimensional words, and it is studied in detail by Amir and Benson [2, 3]. The authors in [8] discuss several generalizations of the familiar Lyndon-Schützenberger periodicity theorem for two-dimensional words.

In this paper, we extend the definition of ℓ-Rauzy graphs from one-dimensional words to two-dimensional words (also known as arrays). An $\ell \times \ell'$-Rauzy graph of order $k \times k'$ for an infinite array or a factorial language F that has arrays is defined, and some examples are given in Section 3. In Section 4, we discuss some structural properties of $\ell \times \ell'$-Rauzy graphs for any infinite periodic array, that includes indegree and outdegree of vertices, the number of components in the graph and the structure of those components. In section 5, we obtain a reduction of $\ell \times \ell'$-Rauzy graph of order $k \times k'$ for infinite periodic arrays to a lower order. In section 6, we prove that $\ell \times \ell'$-Rauzy graph of order $k \times k'$ for the infinite Fibonacci array is strongly connected.

2. Preliminaries

Some basic notations are defined in this section and for more details, one may refer [4, 5, 8, 9].

Let $\Sigma = \{a_1, \cdots, a_n\}$ be a finite alphabet. An array or a two-dimensional word x of size $m \times n$ is of the form

$$x = \begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{matrix}$$

and is denoted by $x[(1,1) \to (m,n)]$ or $x[(1,1); m \times n]$, where $x_{i,j} \in \Sigma, \forall 1 \leq i \leq m, 1 \leq j \leq n$. In $x[(1,1); m \times n]$, $m \times n$ denotes the size of the array.

An array $y = x[(i,i') \to (j,j')]$ is said to be a subword or subarray of x, whose size is $(j - i + 1) \times (j' - i' + 1)$ where $1 \leq i \leq j \leq m$, $1 \leq i' \leq j' \leq n$. An array y is said to be a prefix (or a suffix) of x if $i = 1$ and $i' = 1$ (or $j = m$ and $j' = n$ resp.), denoted by $pre.(x)$ (or $suf.(x)$ resp.).

$\Sigma^{*\times *}$ denotes the set of all arrays, and $\Sigma^{m\times n}$ denotes the set of all arrays of size $m \times n$. If $x \in \Sigma^{m\times n}$, then $x^{k\times \ell}$ is an array in which k copies of x are in each column and ℓ copies of x are in each row whose size is $km \times n\ell$.

An array x of the form $x = y^{m\times n}$ with either $m > 1$ or $n > 1$, is said to be periodic. An array x is said to be primitive if it is not a power of any array. An infinite periodic array $x = y^{\omega\times\omega}$ is defined by repeating the array y infinite times in every row and every column. The period of an infinite periodic array $x = y^{\omega\times\omega}$(for a primitive y) is defined as the size of y.

Let h_m be the mth Fibonacci word, where $h_0 = b,\ h_1 = a,\ h_m = h_{m-1}h_{m-2},\ m \geq 2$. The words h_m are referred to as the finite Fibonacci words. Let F_m be the m^{th} Fibonacci number, where $|h_m| = F_m$. The limit $g_1 = \lim\limits_{m\to\infty} h_m$ is called the infinite Fibonacci word. The infinite Fibonacci word is given by $g_1 = abaababaabaababaab\ldots$, whose m^{th} letter is b (*resp.*, a) if $\lfloor (m+1)\tau \rfloor - \lfloor m\tau \rfloor = a$ (*resp.*, b), where $\tau = \frac{\sqrt{5}-1}{2}, n \geq 1$. Let $g_2 = cdccdcdccdccdcdccd\cdots$, be another infinite Fibonacci word on an alphabet $\{c, d\}$. An infinite Fibonacci array can be defined as $f = [g_1, g_2, g_1, g_1, g_2, g_1, g_2, g_1, g_1, g_2, \cdots]^T$, where $[x, y]^T = \begin{matrix} x \\ y \end{matrix}$ for any two one-dimensional words x, y on Σ.

A set $F \subseteq \Sigma^{*\times *}$ is said to be factorial language if it has all subarrays of the arrays in it, and the factorial language for an infinite array w is the language of subarrays of w, denoted by F_w. Also, $F(k \times k') = \Sigma^{k\times k'} \cap F$, i.e. the set of all arrays in the factorial language F whose size is $k \times k'$.

A labeled polyomino or a brick is a mapping $x : A \to \Sigma'$, where A is a finite subset of $\mathbb{Z} \times \mathbb{Z}$ and Σ' is a finite alphabet.

For example, $x : \{(1,1), (1,2), (2,2), (2,3)\} \to \{a, b, c, d\}$ is defined by

$$x : \begin{cases} (1,1) \to b, \\ (1,2) \to a, \\ (2,2) \to c, \\ (2,3) \to d, \end{cases}$$

and the brick is $\begin{matrix} b & a & \\ & c & d \end{matrix}$.

A directed graph D has a non empty vertex set V and an arc set E. An arc $e = (u, v) \in E$ is an ordered pair of vertices u and v, also written as $e = uv$. In an arc $e = (u, v)$, we say that the arc e leaves the vertex u and enters the vertex v. The indegree/ outdegree of a vertex v is defined as the number of arcs entering/ leaving v, denoted by $deg_{in}(v)/\ deg_{out}(v)$. For any vertex v, the arc $(v, v) \in E$ is known as the self loop.

3. $\ell \times \ell'$-Rauzy Graphs for Infinite Arrays

We extend the idea of ℓ-$\mathbb{R}_w(m)$ in [11] from one-dimensional words to two-dimensional words.

For any two arrays $u, v \in F(k \times k')$, if the $suf.(u)$ of size $\ell \times \ell'$ is same as the $pre.(v)$ of size $\ell \times \ell'$ then the brick formed by the arrays u, v is represented as

$$\begin{array}{ccccccccc}
u_{1,1} & \cdots & u_{1,k'-\ell'+1} & \cdots & u_{1,k'} & & & \\
\vdots & \cdots & \vdots & \cdots & \vdots & & & \\
u_{k-\ell,1} & \cdots & u_{k-\ell,k'-\ell'+1} & \cdots & u_{k-\ell,k'} & & & \\
u_{k-\ell+1,1} & \cdots & \mathbf{u_{k-\ell+1,k'-\ell'+1}} & \cdots & \mathbf{u_{k-\ell+1,k'}} & v_{1,\ell'+1} & \cdots & v_{1,k'} \\
\vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
u_{k,1} & \cdots & \mathbf{u_{k,k'-\ell'+1}} & \cdots & \mathbf{u_{k,k'}} & v_{\ell,\ell'+1} & \cdots & v_{\ell,k'} \\
 & & v_{\ell+1,1} & \cdots & v_{\ell+1,\ell'} & v_{\ell+1,\ell'+1} & \cdots & v_{\ell+1,k'} \\
 & & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
 & & v_{k,1} & \cdots & v_{k,\ell'} & v_{k,\ell'+1} & \cdots & v_{k,k'}
\end{array}$$

and denoted by $b_{u,v}$.

Definition 1. *An $\ell \times \ell'$-Rauzy graph of order $k \times k'$ for a factorial language F that has arrays (or for an infinite array w), is a directed graph (V, E) where $V = F(k \times k')$ (or $F_w(k \times k')$ resp.) and an ordered pair of vertices $(u, v) \in E$ iff the $suf.(u)$ of size $\ell \times \ell'$ is same as the $pre.(v)$ of size $\ell \times \ell'$*

$$i.e.\ u[(k-\ell+1, k'-\ell'+1) \to (k, k')] = v[(1,1) \to (\ell, \ell')]$$

and an array x of size $2k - \ell \times 2k' - \ell'$ that contains the brick $b_{u,v}$ is also in F (or F_w resp.), i.e., $x \in F(2k - \ell \times 2k' - \ell')$ (or $F_w(2k - \ell \times 2k' - \ell')$ resp.). Rauzy graph for a factorial language that has arrays is denoted by $\ell \times \ell'$-$\mathbb{R}_F(k \times k')$, and that for an infinite array is denoted by $\ell \times \ell'$-$\mathbb{R}_w(k \times k')$.

In this article, we discuss the properties of $\ell \times \ell'$-Rauzy graphs of order $k \times k'$ for infinite periodic arrays and the infinite Fibonacci array. Let us start with some examples.

Example 1. *Let $w = x^{\omega \times \omega}$, where $x = \begin{array}{cccccc} a_1 & a_2 & a_1 & a_2 & a_1 & a_2 \\ a_3 & a_4 & a_3 & a_3 & a_4 & a_3 \\ a_5 & a_5 & a_5 & a_6 & a_6 & a_6 \end{array}$ is an array of size 3×6. The $\ell \times \ell'$-$\mathbb{R}_w(4 \times 7)$ is a directed graph with vertices*

$$v_i = \begin{cases} w[(1,i) \to (4, 6+i)] & \text{for } i = 1 \text{ to } 6 \\ w[(2,i) \to (5, 6+i)] & \text{for } i = 7 \text{ to } 12 \\ w[(3,i) \to (6, 6+i)] & \text{for } i = 13 \text{ to } 18 \end{cases}$$

The graphs of 2×5-$\mathbb{R}_w(4 \times 7)$ and 1×1-$\mathbb{R}_w(4 \times 7)$ are shown in Figure 1.

Example 2. *Let f be the infinite Fibonacci array formed from the alphabet $\{a, b, c, d\}$, as defined in preliminaries. The 1×1-$\mathbb{R}_f(2 \times 2)$ is a directed graph shown in Figure 2 with vertices $u_1 = \begin{array}{cc} a & b \\ c & d \end{array}$, $u_2 = \begin{array}{cc} b & a \\ d & c \end{array}$, $u_3 = \begin{array}{cc} a & a \\ c & c \end{array}$, $u_4 = \begin{array}{cc} c & d \\ a & b \end{array}$, $u_5 = \begin{array}{cc} d & c \\ b & a \end{array}$, $u_6 = \begin{array}{cc} c & c \\ a & a \end{array}$, $u_7 = \begin{array}{cc} a & b \\ a & b \end{array}$, $u_8 = \begin{array}{cc} b & a \\ b & a \end{array}$, $u_9 = \begin{array}{cc} a & a \\ a & a \end{array}$.*

4. $\ell \times \ell'$-$\mathbb{R}_w(k \times k')$ for infinite periodic arrays

In this section, we discuss some structural properties of $\ell \times \ell'$-$\mathbb{R}_{x^{\omega \times \omega}}(k \times k')$ such as the number of components in it and structure of each component.

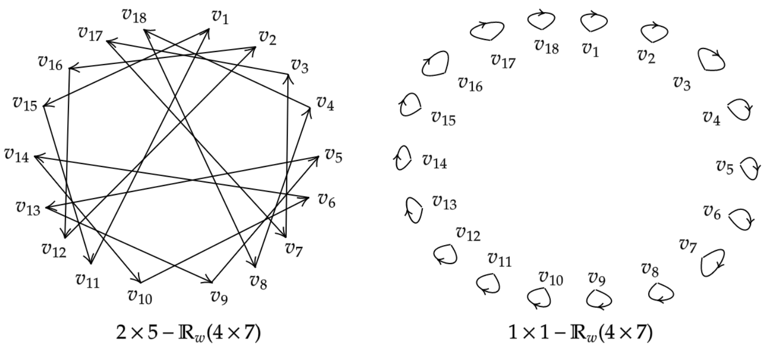

FIGURE 1. $\ell\times\ell'$-$\mathbb{R}_w(4\times 7)$

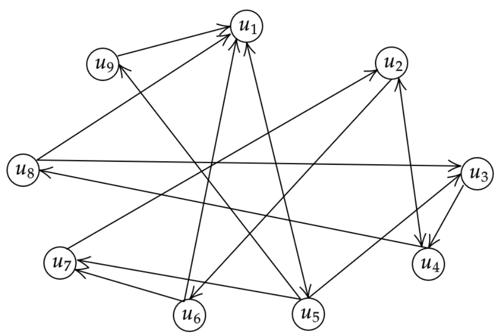

FIGURE 2. 1×1-$\mathbb{R}_f(2\times 2)$

In an array $w = x^{\omega\times\omega}$, we consider x to be a primitive array throughout the section 4 and section 5, as even if x is not primitive, x can be written as $x = y^{r\times r'}$ where y is a primitive array. Also, we consider only the arrays, $x = \begin{matrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{matrix}$ such that no two one-dimensional words $x_{1,1}x_{1,2}\cdots x_{1,n}$, $x_{2,1}x_{2,2}\cdots x_{2,n}, \cdots, x_{m,1}x_{m,2}\cdots x_{m,n}$ are conjugates.

Since x is primitive, the infinite periodic array w has period $m\times n$, and hence,

$$w[(qm+i, pn+j)\to(qm+i', pn+j')] = w[(i,j)\to(i',j')] \tag{1}$$

$1\le i\le i'\le m,\ 1\le j\le j'\le n,\ p,q\in\mathbb{N}$.

The proofs of Proposition 4.1 and Proposition 4.4 are similar to that of, in [10], which can be extended to infinite periodic arrays.

Proposition 4.1. *If $x^{\omega\times\omega}$ is an infinite array of period $m\times n$, then there are mn subarrays of size $k\times k'$, where $k\ge m$ and $k'\ge n$.*

Corollary 4.2. *The number of vertices in an $\ell \times \ell'$-Rauzy graph of order $k \times k'$ is mn, where $k \geq m$ and $k' \geq n$.*

Corollary 4.3. *The number of arcs in an $\ell \times \ell'$-Rauzy graph of order $k \times k'$ is mn, where $k \geq m$ and $k' \geq n$.*

Proposition 4.4. *Let $w = x^{\omega \times \omega}$ be an infinite array of period $m \times n$, where $m, n \in \mathbb{N}$. For any $v \in V(\ell \times \ell'\text{-}\mathbb{R}_w(k \times k'))$, $deg_{out}(v) \geq 1$ and $deg_{in}(v) \geq 1$.*

By Corollary 4.2, Corollary 4.3 and Proposition 4.4, we state the following coroallaries.

Corollary 4.5. *Let $w = x^{\omega \times \omega}$. If $k \geq m$ and $k' \geq n$, then for each $v \in V(\ell \times \ell'\text{-}\mathbb{R}_w(k \times k'))$, $deg_{in}(v) = deg_{out}(v) = 1$.*

Corollary 4.6. *Let $w = x^{\omega \times \omega}$. If $k \geq m$ and $k' \geq n$, then each component of $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$ is a directed cycle with atleast 2 vertices or a self loop.*

We see that $\ell \times \ell'$-Rauzy graphs have cycles or self loops as a component in it, and now, let us find that for what values of k, k', ℓ, ℓ' it occurs.

Theorem 4.7. *Let $w = x^{\omega \times \omega}$ and size of x is $m \times n$. All the arcs in $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$ are self loops $\iff k = qm + \ell,\ k' = pn + \ell',\ p, q \in \mathbb{N},\ \ell \leq m,\ \ell' \leq n$.*

Proof. Let us assume that all the arcs in $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$ are self loops, and suppose $k \neq qm + \ell$ or $k' \neq pn + \ell'$. For every vertex $v = w[(i, j) \to (i+k-1, j+k'-1)]$, there always exists a vertex $v' = w[(i+k-\ell, j+k'-\ell') \to (i+2k-\ell-1, j+2k'-\ell'-1)]$ such that vv' forms an arc. This is a contradiction to the assumption that all the arcs are self loops.

Conversely, if $k = qm + \ell,\ k' = pn + \ell',\ p, q \in \mathbb{N}$ for $\ell \leq m,\ \ell' \leq n$, then the suffix array of size $\ell \times \ell'$ is same as the prefix array of size $\ell \times \ell'$ in any vertex v. So, an arc that leaves the vertex v_i enters the same vertex.

Hence, all the arcs in $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$ are self loops $\iff k = qm + \ell$ and $k' = pn + \ell'$. ☐

The graph of $1 \times 1\text{-}\mathbb{R}_w(4 \times 7)$ in Figure 1 illustrates Theorem 4.7 that all the arcs in the graph are self loops.

Next, we find the number of cycles in any $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$, and length of each cycle.

Theorem 4.8. *Let $w = x^{\omega \times \omega}$ and the size of x is $m \times n$. For $k \neq qm+\ell,\ k' \neq pn+\ell'$, $q, p \in \mathbb{N},\ k \geq m, k' \geq n$, the $\ell \times \ell'$-Rauzy graph of order $k \times k'$ has α-components and each component is a β-cycle, where*

(1) $\alpha = \frac{mn}{\beta}$ *and* $\beta = lcm(\frac{lcm(k'-\ell', n)}{k'-\ell'},\ \frac{lcm(k-\ell, m)}{k-\ell})$ *for* $\ell \neq k$ *and* $\ell' \neq k'$.

(2) $\alpha = m \cdot gcd(k' - \ell', n)$ *and* $\beta = \frac{lcm(k'-\ell', n)}{k'-\ell'}$ *for* $\ell = k$ *and* $\ell' \neq k'$.

(3) $\alpha = n \cdot gcd(k - \ell, m)$ *and* $\beta = \frac{lcm(k-\ell, m)}{k-\ell}$ *for* $\ell' = k'$ *and* $\ell \neq k$.

Proof. By Corollary 4.6, the graph $\ell \times \ell'\text{-}\mathbb{R}_w(k \times k')$ contains cycles with atleast two vertices or a self loop. Given $k \neq qm + \ell$ and $k' \neq pn + \ell'$, the existence of self loop is eliminated by Theorem 4.7, and so it contains only cycles with atleast two vertices. Let us choose a vertex arbitrarily, (say v_1) and it lies on a cycle with atleast two vertices.

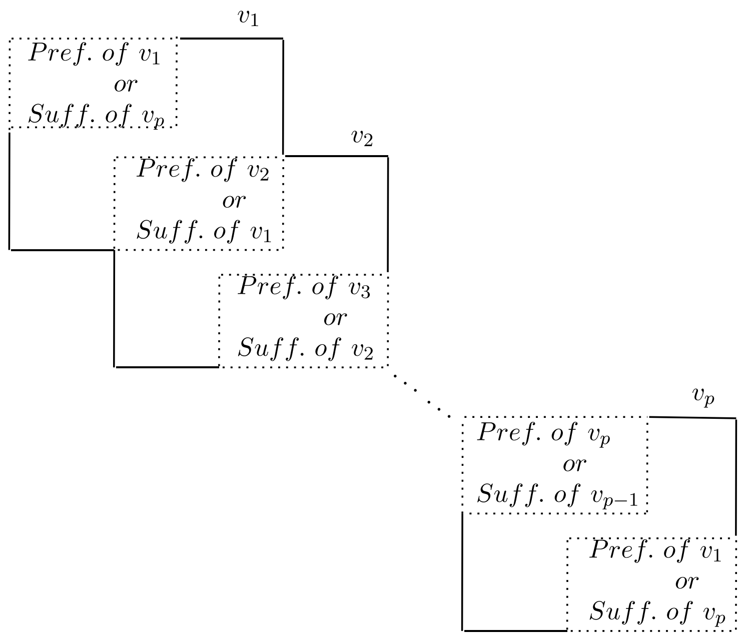

FIGURE 3. The brick formed by a cycle $C:\ v_1v_2\cdots v_pv_1$

The array associated with a cycle $v_1v_2\cdots v_pv_1$ is the smallest array that contains the brick formed by the cycle $v_1v_2\cdots v_pv_1$. The brick formed by a cycle $C : v_1v_2\cdots v_pv_1$ is shown in the Figure 3. Consider the brick formed by the cycle $C:\ v_1v_2\cdots v_pv_1$. Let us count the number of columns in this brick. There are k' columns in the array of the vertex v_1. The prefix of size $\ell \times \ell'$ of the vertex v_2 is shared with first vertex v_1, and so there are only $k' - \ell'$ columns to be counted in v_2. Likewise, there are only $k' - \ell'$ columns to be counted from v_2 to v_{p-1}. In the vertex v_p, the counting differs for $\ell' < \frac{k'}{2}$, $\ell' > \frac{k'}{2}$, and $\ell' = \frac{k'}{2}$.

First, let us consider that $\ell' < \frac{k'}{2}$. There are ℓ' columns of the prefix word of size $\ell \times \ell'$ of v_p counted in v_{p-1}, and ℓ' columns of the suffix word of size $\ell \times \ell'$ of v_p coincides with v_1. So, there are $k' - 2\ell'$ columns yet to be counted in v_p. Hence, there are $k' + (p-2)(k'-\ell') + k' - 2\ell' = p(k'-\ell')$ columns in the brick. In a similar way for $\ell' > \frac{k'}{2}$ and $\ell' = \frac{k'}{2}$, one can count the number of columns in v_p, and find that the number of columns in the brick formed by C is $p(k'-\ell')$.

The number of columns in a brick formed by the cycle C is a multiple of $k' - \ell'$ in all the cases. As v_p makes an arc with vertex v_1, the number of columns is also a multiple of n (the number of columns in x) in all the cases.

The number of rows in a brick formed by a cycle C can be counted in a similar way, and we see that it is $p(k-\ell)$. As v_p makes an arc with vertex v_1, the number of rows is also a multiple of m (the number of rows in x) in all the cases $\ell < \frac{k}{2}$, $\ell > \frac{k}{2}$ and $\ell = \frac{k}{2}$.

Proof of 1. The number of columns in a brick formed by a cycle is a multiple of $k' - \ell'$, and also a multiple of n in all the cases. Thus, the multiple of $lcm(k'-\ell', n)$

is the required number of columns in the smallest array that contains the brick formed by the cycle C. We observed that the number of the columns is $p(k'-\ell')$ i.e. multiple of $lcm(k'-\ell',n)=p(k'-\ell')$. Hence, the number of vertices p in cycle C is a multiple of $\frac{lcm(k'-\ell',n)}{(k'-\ell')}$.

The number of rows in a brick formed by a cycle is a multiple of $k-\ell$, and also a multiple of m. Thus, the multiple of $lcm(k-\ell,m)$ is the required number of rows in the smallest array that contains the brick formed by the cycle C. And, we have seen that the number of the rows is $p(k-\ell)$ i.e. multiple of $lcm(k-\ell,m)=p(k-\ell)$. Hence, the number of vertices in a cycle is a multiple of $\frac{lcm(k-\ell,m)}{(k-\ell)}$.

Here, the number of vertices is a multiple of $\frac{lcm(k'-\ell',n)}{(k'-\ell')}$ as well as $\frac{lcm(k-\ell,m)}{(k-\ell)}$. Thus, the least common multiple of $\frac{lcm(k'-\ell',n)}{k'-\ell'}$ and $\frac{lcm(k-\ell,m)}{k-\ell}$ is the required number of vertices to form a cycle C.

We chose a vertex arbitrarily and the component containing the vertex is $l.c.m$ $(\frac{lcm(k'-\ell',n)}{k'-\ell'},\ \frac{lcm(k-\ell,m)}{k-\ell})$-cycle. Thus, each component in $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ is a β-cycle, where $\beta=l.c.m(\frac{lcm(k'-\ell',n)}{k'-\ell'},\ \frac{lcm(k-\ell,m)}{k-\ell})$ for $\ell\neq k$ and $\ell'\neq k'$. And, the number of components is

$$\frac{no.\ of\ vertices\ in\ the\ graph}{no.\ of\ vertices\ in\ each\ component}=\frac{mn}{l.c.m(\frac{lcm(k'-\ell',n)}{k'-\ell'},\ \frac{lcm(k-\ell,m)}{k-\ell})}.$$

Thus, $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ has α-components for $\ell\neq k$ and $\ell'\neq k'$, where $\alpha=\frac{mn}{\beta}$.

Other two cases can be proved, similarly. □

The graph of 2×5-$\mathbb{R}_w(4\times 7)$ in Figure 1 illustrates Theorem 4.8.

Corollary 4.9. *Let $w=x^{\omega\times\omega}$. For $k\neq qm+\ell,\ k'\neq pn+\ell',\ q,p\in\mathbb{N}$ and $\beta=mn$, the graph of $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ is strongly connected.*

For example, the 2×3-$\mathbb{R}_w(3\times 5)$ for an array $w=x^{\omega\times\omega}$ is a strongly connected graph, where size of x is also 3×5. By theorem 4.8, the length of the directed cycle $\beta=lcm(\frac{lcm(2,5)}{2},\ \frac{lcm(1,3)}{1})=lcm(5,3)=15$.

5. Reduction of $\ell\times\ell'$-Rauzy graphs of order $k\times k'$ for $x^{\omega\times\omega}$

In this section, we see that for any $k>m, k'>n$, $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ can be reduced to a lower order or can be studied from some lower order $\ell\times\ell'$-Rauzy graph itself.

Theorem 5.1. *Let $w=x^{\omega\times\omega}$. If $\ell_1=im+\ell,\ \ell'_1=i'n+\ell'$ then $\ell_1\times\ell'_1$-$\mathbb{R}_w(k\times k')\simeq\ell\times\ell'$-$\mathbb{R}_w(k\times k')$.*

Proof. The vertex sets $V(\ell_1\times\ell'_1$-$\mathbb{R}_w(k\times k'))=V(\ell\times\ell'$-$\mathbb{R}_w(k\times k'))=F_w(k\times k')$. Let us consider a vertex $u=w[(j,j')\to(k+j-1,k'+j'-1)]$ in $\ell_1\times\ell'_1$-$\mathbb{R}_w(k\times k')$ and it forms an arc $e'=(u,v)$ with $v=w[(j+k-(im+\ell),j'+k'-(i'n+\ell'))\to(k+j-(im+\ell)+k-1,k'+j'-(i'n+\ell')+k'-1)](=w[(j+k-\ell,j'+k'-\ell')\to(j+2k-\ell-1,j'+2k'-\ell'-1]$ by equation (1)). Also, $e=(u,v)$ forms an arc in $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ by definition. Hence, $\ell_1\times\ell'_1$-$\mathbb{R}_w(k\times k')\simeq\ell\times\ell'$-$\mathbb{R}_w(k\times k')$. □

Theorem 5.2. *Let $w=x^{\omega\times\omega}$ and $k>m,\ k'>n$. Then for some $i,i',j,j'\geq 1$, the $\ell\times\ell'$-Rauzy graph of order $k\times k'$, i.e., $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$*

$$\simeq \begin{cases} (\ell-(k-m))\times(\ell'-(k'-n))\text{-}\mathbb{R}_w(m\times n) & if\ \ell>k-m,\ \ell'>k'-n \\ ((i+1)m+\ell-k)\times((i'+1)n+\ell'-k')\text{-}\mathbb{R}_w(m\times n) & if\ \ell<k-m,\ \ell\neq k-jm, \\ & \&\ \ell'<k'-n,\ \ell'\neq k'-j'n \\ (k-m)\times(k'-n)\text{-}\mathbb{R}_w(m\times n) & if\ \ell=k-jm,\ \ell=k'-j'n \\ (\ell-(k-m))\times((i'+1)n+\ell'-k')\text{-}\mathbb{R}_w(m\times n) & if\ \ell>k-m, \\ & \&\ \ell'<k'-n,\ \ell'\neq k'-j'n \\ ((i+1)m+\ell-k)\times(\ell'-(k'-n))\text{-}\mathbb{R}_w(m\times n) & if\ \ell<k-m,\ \ell\neq k-jm, \\ & \&\ \ell'>k'-n \end{cases}$$

Proof. **Case 1:** Let $\ell > k-m,\ \ell' > k'-n$. We know $\ell < k, \ell' < k'$ in $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$. Here, $\ell-k<0 \Rightarrow \ell-k+m<m$ and similarly, $\ell'-k'+n<n$. Hence, sharing an array of size $(\ell-k+m)\times(\ell'-k'+n)$ is possible. Now by Theorem 4.8, each component in $(\ell-(k-m))\times(\ell'-(k'-n))$-$\mathbb{R}_w(m\times n)$ is a cycle of length $\beta = lcm(\frac{lcm(n-(\ell'-k'+n),n)}{k'-\ell'},\ \frac{lcm(m-(\ell-k+m),m)}{k-\ell}) = lcm(\frac{lcm(k'-\ell',n)}{k'-\ell'},\ \frac{lcm(k-\ell,m)}{k-\ell})$, and has $\alpha = \frac{mn}{\beta}$ components, which are same as in $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$.

We have shown that both $(\ell-(k-m))\times(\ell'-(k'-n))$-$\mathbb{R}_w(m\times n)$ and $\ell\times\ell'$-$\mathbb{R}_w(k\times k')$ has α number of cycles that are of length β. Thus for $\ell>k-m,\ \ell'>k'-n$, $(\ell-(k-m))\times(\ell'-(k'-n))$-$\mathbb{R}_w(m\times n) \simeq \ell\times\ell'$-$\mathbb{R}_w(k\times k')$.

Case 2: Let $\ell<k-m,\ \ell\neq k-jm,\ \ell'<k'-n,\ \ell'\neq k'-j'n$. If $k-2m<\ell<k-m$, then $k-m<m+\ell<k$. If $k-3m<\ell<k-2m$, then $k-m<2m+\ell<k$. And going on like this, in general if $k-(j+1)m<\ell<k-jm$, then $k-m<jm+\ell<k$. Now, it is clear to conclude that if $\ell<k-m$ there always exists an i such that $k-m<im+\ell<k$. Similarly, if $\ell'<k'-n$ there always exists an i' such that $k'-n<i'n+\ell'<k'$.

Now by theorem 5.1, $\ell\times\ell'$-$\mathbb{R}_w(k\times k') \simeq (im+\ell)\times(i'n+\ell')$-$\mathbb{R}_w(k\times k')$. Here, $im+\ell<k \Rightarrow \ell-k+im<0 \Rightarrow \ell-k+(i+1)m<m$ and $i'n+\ell'<k' \Rightarrow \ell'-k'+(i'+1)n<n$. Hence, sharing an array of size $(\ell-k+(i+1)m)\times\ (\ell'-k'+(i'+1)n)$ is possible. We saw that $im+\ell>k-m,\ i'n+\ell'>k'-n$ and by first case of the theorem,

$$(im+\ell)\times(i'n+\ell')\text{-}\mathbb{R}_w(k\times k') \simeq ((i+1)m+\ell-k)\times((i'+1)n+\ell'-k')\text{-}\mathbb{R}_w(m\times n).$$

Case 3: Let $\ell=k-jm,\ \ell'=k'-j'n$. By theorem 4.7, $(k-jm)\times(k'-j'n)$-$\mathbb{R}_w(k\times k')$ has only self loops for given j,j'. As there are mn vertices in $(k-jm)\times(k'-j'n)$-$\mathbb{R}_w(k\times k')$, there are mn self loops in it. Also, $(k-m)\times(k'-n)$-$\mathbb{R}_w(m\times n)$ has mn self loops in it, by definition. Hence,

$$\ell\times\ell'\text{-}\mathbb{R}_w(k\times k') \simeq (k-m)\times(k'-n)\text{-}\mathbb{R}_w(m\times n)$$

For other cases of ℓ and ℓ', it can be proved in a similar way. □

6. $\ell\times\ell'$-Rauzy graphs for the Infinite Fibonacci array

In this section, we prove that the $\ell\times\ell'$-Rauzy graph of order $k\times k'$ for the infinite Fibonacci array is strongly connected for $1\leq\ell<k,\ 1\leq\ell'<k'$.

Theorem 6.1. *For a given $k,k',\ell,\ell'\in\mathbb{N}$ such that $1\leq\ell<k,\ 1\leq\ell'<k'$, the $\ell\times\ell'$-Rauzy graph of infinite Fibonacci array f of order $k\times k'$ is strongly connected.*

Proof. For a given $k,k',\ell,\ell'\in\mathbb{N}$ such that $1\leq\ell<k,\ 1\leq\ell'<k'$, the distinct subarrays of size $k\times k'$ of infinite Fibonacci array f is the set of all vertices in

$\ell \times \ell'$-$\mathbb{R}_f(k \times k')$. It is well known that the number of subwords of infinite Fibonacci one-dimensional word of length k is $k+1$. Thus, it is easy see that the number of subarrays of the infinite Fibonacci array of size $k \times k'$ is $(k+1)(k'+1)$. Let the vertices of $\ell \times \ell'$-$\mathbb{R}_f(k \times k')$ be $v_{1,1}, \cdots, v_{1,k'+1}, v_{2,1}, \cdots, v_{2,k'+1}, \cdots, v_{k+1,1}, \cdots, v_{k+1,k'+1}$.

For some given k, k', choose m, n such that $F_m \le k \le F_{m+1}, F_n \le k' \le F_{n+1}$. From Proposition 2.7 in [6], the first occurrences of $(k+1)(k'+1)$ distinct subarrays of size $k \times k'$ are given by

$$v_{j,j'} = \begin{cases} f[(j,j');\ k \times k'] & \text{if } \ 1 \le j \le F_m,\ 1 \le j' \le F_n \\ f[(j + F_{m+1} - (k+1), j' + F_{n+1} - (k'+1)); k \times k'] & \\ \qquad \text{if } \ F_m + 1 \le j \le k+1,\ F_n + 1 \le j' \le k'+1 & \\ f[(j, j' + F_{n+1} - k' - 1); k \times k'] & \text{if } 1 \le j \le F_m,\ F_n + 1 \le j' \le k'+1 \\ f[(j + F_{m+1} - k - 1, j'); k \times k'] & \text{if } F_m + 1 \le j \le k+1,\ 1 \le j' \le F_n \end{cases}$$

Using Corollary 3.6 and Proposition 3.9 in [6], all the locations of $v_{j,j'}$ in the infinite Fibonacci array are given by

$$loc.(v_{j,j'}) = \begin{cases} \{(tF_{m-1} + \lfloor (t+1)\tau \rfloor F_{m-2} + j,\ t'F_{n-1} + \lfloor (t'+1)\tau \rfloor F_{n-2} + j')\} \\ \qquad \text{if } \ 1 \le j \le F_{m+1} - k - 1,\ 1 \le j' \le F_{n+1} - k' - 1 \\ \{(tF_m + \lfloor (t+1)\tau \rfloor F_{m-1} + j,\ t'F_n + \lfloor (t'+1)\tau \rfloor F_{n-1} + j')\} \\ \qquad \text{if } \ F_{m+1} - k \le j \le F_m,\ F_{n+1} - k' \le j' \le F_n \\ \{(tF_{m+1} + \lfloor (t+1)\tau \rfloor F_m + j + F_{m+1} - (k+1), \\ \qquad t'F_{n+1} + \lfloor (t'+1)\tau \rfloor F_n + j' + F_{n+1} - (k'+1))\} \\ \qquad \text{if } \ F_m + 1 \le j \le k+1,\ F_n + 1 \le j' \le k'+1 \\ \{(tF_{m-1} + \lfloor (t+1)\tau \rfloor F_{m-2} + j,\ t'F_n + \lfloor (t'+1)\tau \rfloor F_{n-1} + j')\} \\ \qquad \text{if } \ 1 \le j \le F_{m+1} - k - 1,\ F_{n+1} - k' \le j' \le F_n \\ \{(tF_{m-1} + \lfloor (t+1)\tau \rfloor F_{m-2} + j, \\ \qquad t'F_{n+1} + \lfloor (t'+1)\tau \rfloor F_n + j' + F_{n+1} - (k'+1))\} \\ \qquad \text{if } \ 1 \le j \le F_{m+1} - k - 1,\ F_n + 1 \le j' \le k'+1 \\ \{(tF_m + \lfloor (t+1)\tau \rfloor F_{m-1} + j,\ t'F_{n-1} + \lfloor (t'+1)\tau \rfloor F_{n-2} + j')\} \\ \qquad \text{if } \ F_{m+1} - k \le j \le F_m,\ 1 \le j' \le F_{n+1} - k' - 1 \\ \{(tF_m + \lfloor (t+1)\tau \rfloor F_{m-1} + j, \\ \qquad t'F_{n+1} + \lfloor (t'+1)\tau \rfloor F_n + j' + F_{n+1} - (k'+1))\} \\ \qquad \text{if } \ F_{m+1} - k \le j \le F_m,\ F_n + 1 \le j' \le k'+1 \\ \{(tF_{m+1} + \lfloor (t+1)\tau \rfloor F_m + j + F_{m+1} - (k+1), \\ \qquad t'F_{n-1} + \lfloor (t'+1)\tau \rfloor F_{n-2} + j')\} \\ \qquad \text{if } \ F_m + 1 \le j \le k+1,\ 1 \le j' \le F_{n+1} - k' - 1 \\ \{(tF_{m+1} + \lfloor (t+1)\tau \rfloor F_m + j + F_{m+1} - (k+1), \\ \qquad t'F_n + \lfloor (t'+1)\tau \rfloor F_{n-1} + j')\} \\ \qquad \text{if } \ F_m + 1 \le j \le k+1,\ F_{n+1} - k' \le j' \le F_n \end{cases}$$

where $t, t' \ge 0$ in each of those sets. We see that locations of $v_{j,j'}$ for any j, j' is of the form

$$(\quad bt + c\lfloor (t+1)\tau \rfloor + d, \quad b't' + c'\lfloor (t'+1)\tau \rfloor + d' \quad)$$

where $b,\ c\in\{F_{m-2},\ F_{m-1},\ F_m,\ F_{m+1}\}$, $d\in\{j, j+F_{m+1}-(k+1)\}$, $b',\ c'\in\{F_{n-2},\ F_{n-1},\ F_n,\ F_{n+1}\}$, $d\in\{j', j'+F_{n+1}-(k'+1)\}$.

Let us consider the path (*say* P_1) that starts from the subarray of size $k\times k'$, located in the first position of infinite Fibonacci array. By the definition of $\ell\times\ell'$-Rauzy graphs, the path P_1 is given by

$$f[(1,1);k\times k']\to\cdots\to f[(1+i(k-\ell),1+i'(k'-\ell'));k\times k']\to\cdots$$

In path P_1, it is clear that any subarray of the form $f[(1+i(k-\ell),1+i'(k'-\ell'));k\times k']$ is reachable from $f[(1,1);k\times k']$ or $v_{1,1}$. If atleast one location of each vertex is of the form $(1+i(k-\ell),1+i'(k'-\ell'))$, then every vertex is reachable from $v_{1,1}$.

The integer solution to the equations

$$1+i(k-\ell)=bt+c\lfloor(t+1)\tau\rfloor+d \text{ for each } 1\le j\le k+1$$

$$1+i'(k'-\ell')=b't'+c'\lfloor(t'+1)\tau\rfloor+d' \text{ for each } 1\le j'\le k'+1$$

guarantee that atleast one location of each vertex is of the form $(1+i(k-\ell),1+i'(k'-\ell'))$. Let $x_1=i,\ x_2=t,\ x_3=\lfloor(t+1)\tau\rfloor, y_1=i',\ y_2=t',\ y_3=\lfloor(t'+1)\tau\rfloor$ be the variables. The equations can be rewritten as

$$ax_1-bx_2-cx_3=d-1,\qquad a'y_1-b'y_2-c'y_3=d'-1$$

The linear Diophantine equation $a_1x_1-b_1x_2-c_1x_3=d_1$ has infinite integer solutions $\iff$ $gcd(a_1,b_1,c_1)|d_1$.

It is well known that any two consecutive Fibonacci numbers are co-prime. And, $gcd(a,b,c)=1|d$, $gcd(a',b',c')=1|d'$, for each $1\le j\le k+1,\ 1\le j'\le k'+1$.

Now, it is clear that the equations $ax_1-bx_2-cx_3=d-1,\ a'y_1-b'y_2-c'y_3=d'-1$ has infinite integer solutions for each $1\le j\le k+1,\ 1\le j'\le k'+1$. Thus, every vertex is reachable from $v_{1,1}$ in the path P_1. As every vertex is located infinitely many times in the path P_1, the vertex $v_{1,1}$ is reachable from any other vertex. Hence, $\ell\times\ell'$-$\mathbb{R}_f(k\times k')$ is strongly connected. □

7. Conclusion

The $\ell\times\ell'$-Rauzy graphs for infinite arrays are defined. The properties of $\ell\times\ell'$-Rauzy graphs for infinite periodic arrays and the infinite Fibonacci array are studied in this paper. As locations of subarrays of an infinite array play a vital role in the study of $\ell\times\ell'$-Rauzy graphs, and is not yet known for many special words, the study becomes tougher and interesting to explore much. Further, the study on special two-dimensional words such as Thue-morse array would be interesting.

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